Measuring Chaos from Spatial Information

Journal club

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Solé, R. V., & Bascompte, J. (1995). Measuring chaos from spatial information. *Journal of Theoretical Biology*, *175*(2), 139–147. https://doi.org/10.1006/jtbi.1995.0126

Introduction

- In most of the nonlinear system, detecting chaos is always one of the most important topics.

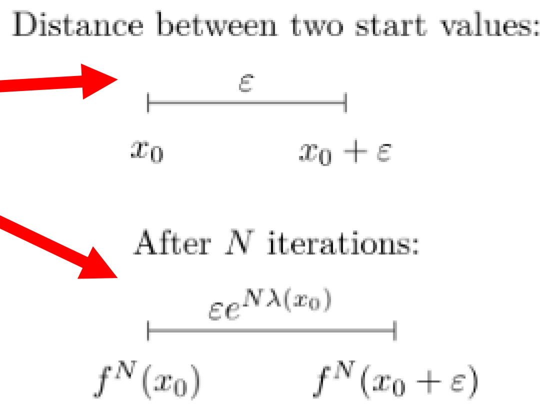
- Conventional method is to use Lyapunov exponent (LE) in time scale.
- Basically, run the simulation/experiment for quite a long time, and record the LE

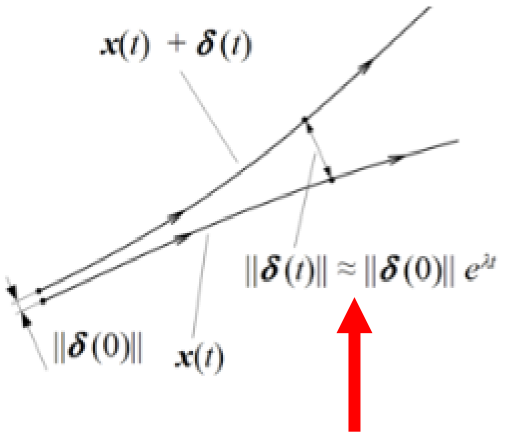
- However, some system is not able to have data with that many time steps (E.g., some ecology system can be affected by transitions between attractors when running time is long)
- So here, it introduces new method to get LE using spatiotemporal information [1]

Introduction – what is Lyapunov exponent

- Lyapunov exponent of a dynamical system is to characterizes the rate of separation of infinitesimally close trajectories. [2]
- Formula for LE [3] (Eq 2a in paper):

$$\lambda = \lim_{t \to \infty} \lim_{|\delta \mathbf{Z}_0| \to 0} \frac{1}{t} \ln \frac{|\delta \mathbf{Z}(t)|}{|\delta \mathbf{Z}_0|}$$





This means chaos

- $\lambda > 0$: chaotic system
  - δZ(t)>>δZ(0) (difference has exp growth)
- $\lambda = 0$: periodic
  - δZ(t)=δZ(0) (difference has no growth)
- $\lambda < 0$: stable (convergent to fix point)

Introduction – what is Lyapunov exponent

- There is another formula for LE, which cannot apply here
- Another formula for LE (Eq 1 in paper):

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|$$

- This one needs to know the exact motional expression (since we need to get derivative) of dynamical system
- But for most of the cases, we have only a temporal series recorded, usually without information about mechanism behind

Introduction – what is Lyapunov exponent

- So, it applies Eq below for conventional method

$$\lambda = \lim_{t \to \infty} \lim_{|\delta \mathbf{Z}_0| \to 0} \frac{1}{t} \ln \frac{|\delta \mathbf{Z}(t)|}{|\delta \mathbf{Z}_0|}$$

- Now, as we can see, it requires a long time series (usually O(1000) data points) and so it cannot be applied to current data without a long temporal scale.

- And this paper introduces a new one.

Method – a new spatiotemporal way

- Coupled map lattices (CML) is used to prove this method
- Every point in CML is just like logistic map, but also affected by its neighboring (Laplacian):

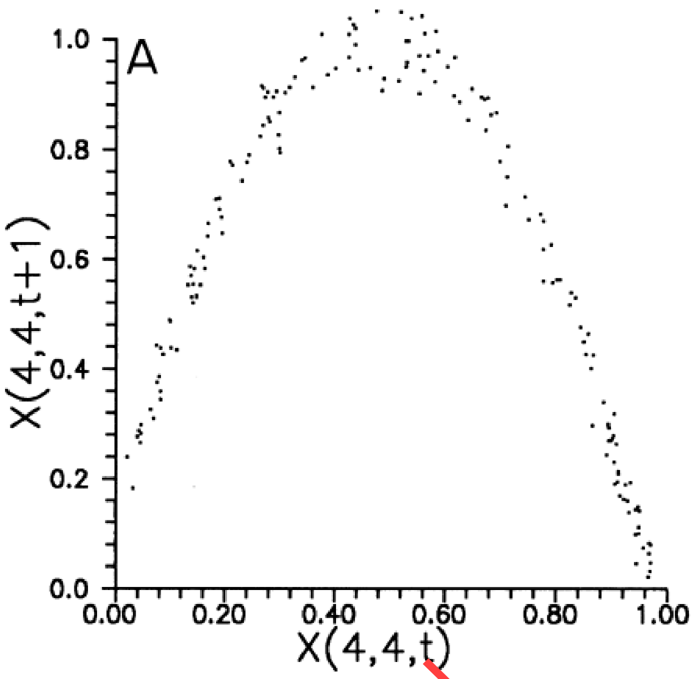
$$x_{n+1}(\mathbf{k}) = \mu x_n(\mathbf{k})(1 - x_n(\mathbf{k})) + D\nabla^2 x_n(r),$$

- Now, in this 2D lattice map, it runs twice, first time recording one point with long time (200 steps)
- Second time recording all points in lattice, with a short time

Method – a new spatiotemporal way

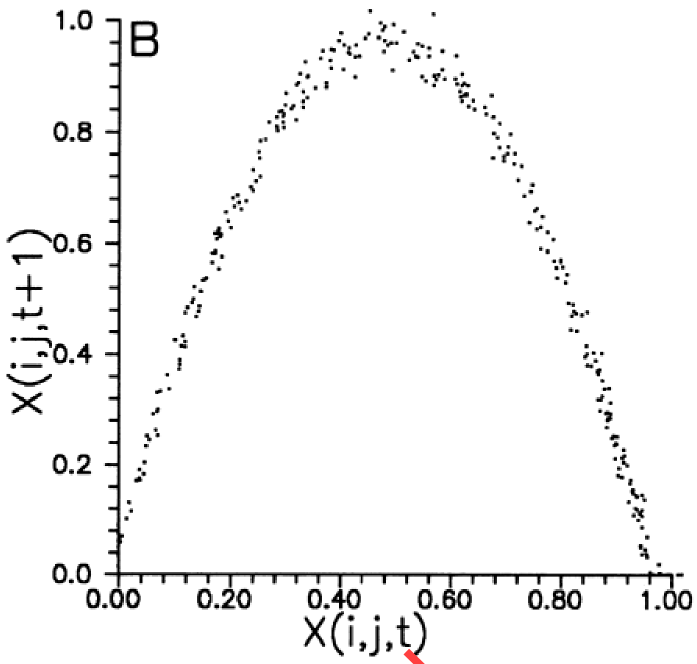
- First time one point with long time; Second time all points with a short time

First, one point only



t = [0,200]

Second, all points



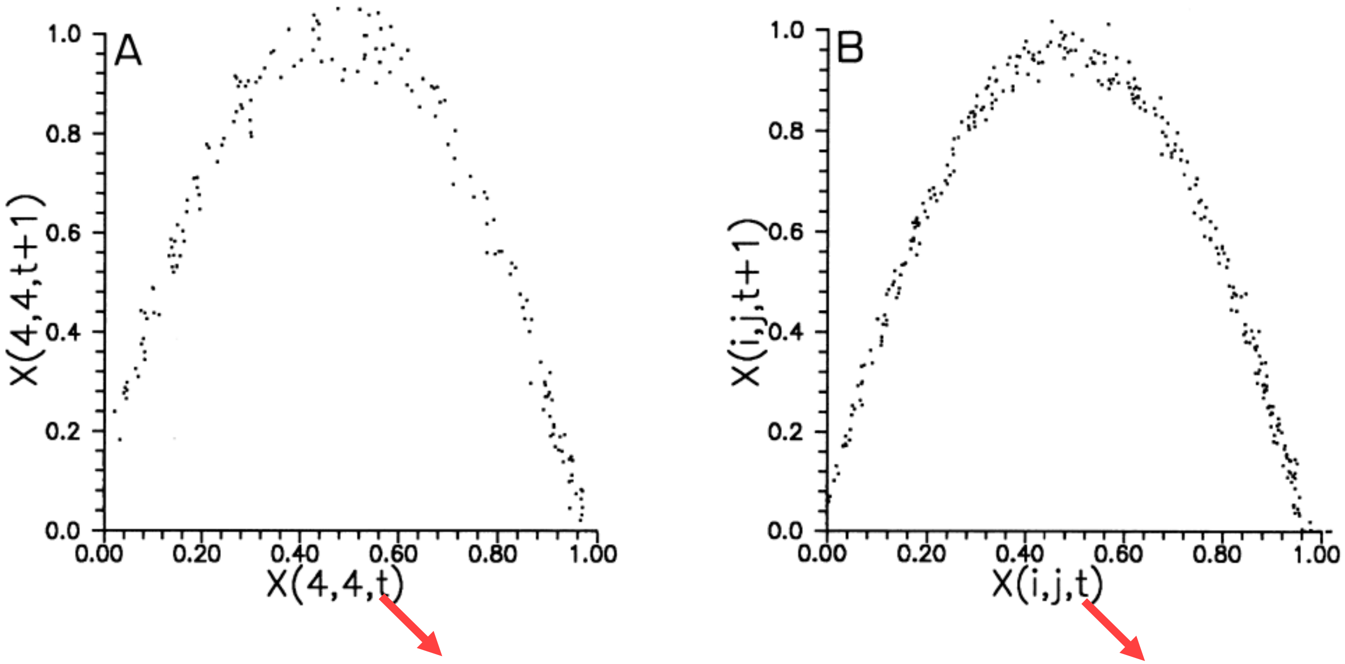
t = [0,10]

Method – a new spatiotemporal way

- Now we see that they are very similar.
- The lack of temporal information is compensated by the spatial information

First, one point only

Second, all points



t = [0,200]

t = [0,10]

Method – a new spatiotemporal way

- The lack of temporal information is compensated by the spatial information
- Then we can also apply it onto LE
- A new spatiotemporal LE is defined:

$$\lambda_s(d) = \frac{1}{N_p} \sum_{i=1}^{m-d} \sum_{\langle \mathbf{k},\mathbf{h} \rangle} Ln\left[\frac{\| \mathbf{X}^j_{i+1}(\mathbf{k}) - \mathbf{X}^j_{i+1}(\mathbf{h}) \|}{\| \mathbf{X}^j_i(\mathbf{k}) - \mathbf{X}^j_i(\mathbf{h}) \|} \right]. \qquad \| \mathbf{X}^j_i(\mathbf{k}) - \mathbf{X}^j_i(\mathbf{h}) \| = \left[\sum_{u=i}^{i+d-1} (x^j_u(\mathbf{k}) - x^j_u(\mathbf{h}))^2 \right]^{1/2} < \epsilon$$

  - s means spatial
  - k = every point of 2D map from t_i to t_(i+d)
  - h = neighboring points of k from t_i to t_(i+d) (neighbor distance < a small value)
  - d = embedded dimension
  - N_p = number of <k, h> pairs (take average)
  - j = type of X variable
  - i = time step

Method – a new spatiotemporal way

- A new spatiotemporal LE is defined
- Compared to the old one:

$$\lambda_s(d) = \frac{1}{N_p} \sum_{i=1}^{m-d} \sum_{\langle \mathbf{k},\mathbf{h} \rangle} Ln\left[\frac{\| \mathbf{X}_{i+1}^{j}(\mathbf{k}) - \mathbf{X}_{i+1}^{j}(\mathbf{h}) \|}{\| \mathbf{X}_{i}^{j}(\mathbf{k}) - \mathbf{X}_{i}^{j}(\mathbf{h}) \|} \right].$$

$$\lambda = \lim_{t \to \infty} \lim_{|\delta \mathbf{Z}_0| \to 0} \frac{1}{t} \ln \frac{|\delta \mathbf{Z}(t)|}{|\delta \mathbf{Z}_0|}$$

- t = [0, m-d]
- Points = all

require less time

but more spatial info

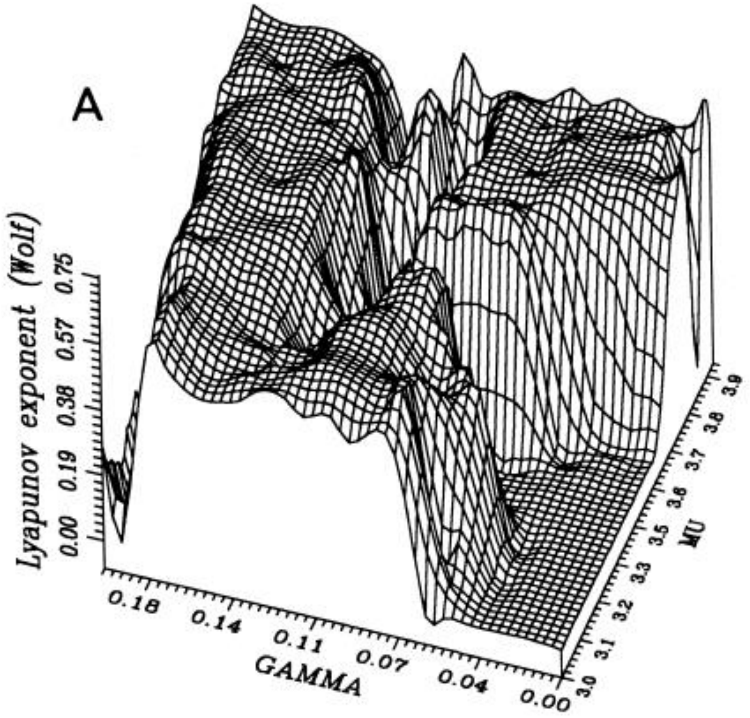
t = [0,inf]

points = one

Results – LE obtained from new method

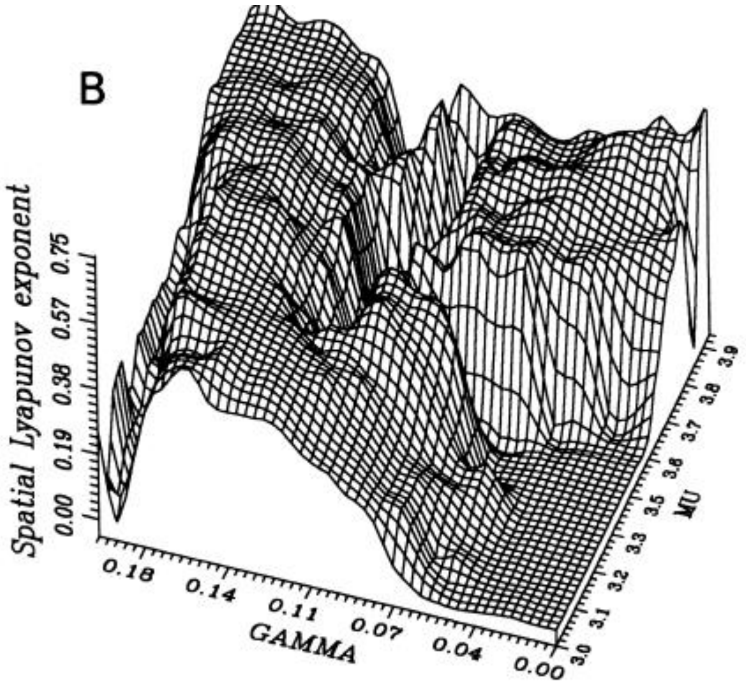
- **1. Logistic CML**

1500 time steps after 1500 transients



Conventional LE, Gamma = D

12 time steps after 1500 transients



New method LE

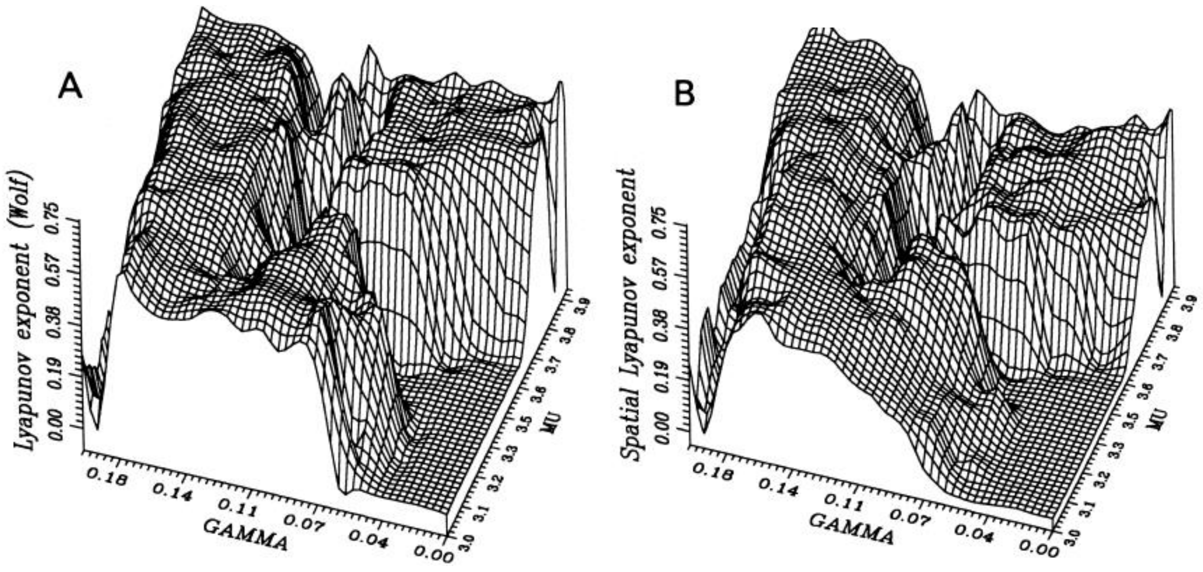
$$x_{n+1}(\mathbf{k}) = \mu x_n(\mathbf{k})(1 - x_n(\mathbf{k})) + D\nabla^2 x_n(r),$$

with D the diffusion rate and

$$\nabla^2 x_n(r) = x_n(i-1, j) + x_n(i+1, j) + x_n(i, j-1) + x_n(i, j+1) - 4x_n(i, j)$$

Results – LE obtained from new method

- The two exponents show the same domains of stable, periodic and chaotic attractors
- For further evidence of the validity, host–parasitoid CML is applied.



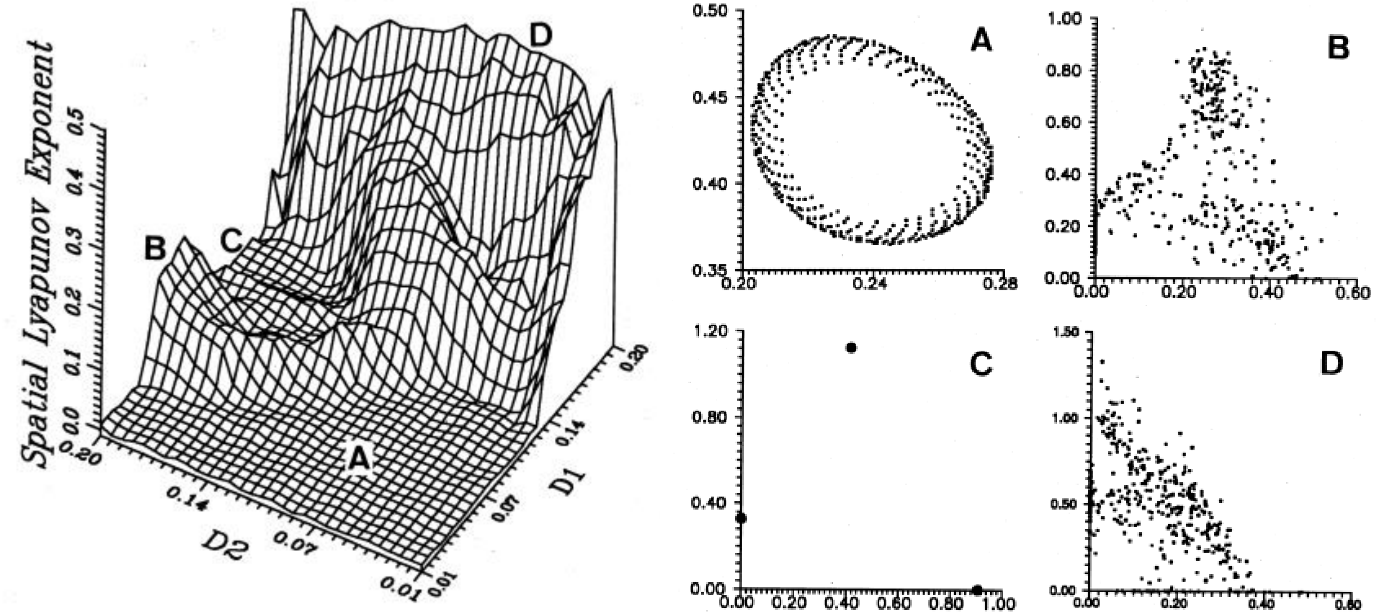
Results – LE obtained from new method

- **2. Host–parasitoid CML.**

$$x_{n+1}(\mathbf{k}) = \mu x_n(\mathbf{k})(1 - x_n(\mathbf{k})) \times \exp(-\beta y_n(\mathbf{k})) + D_1 \nabla^2 x_n(r)$$

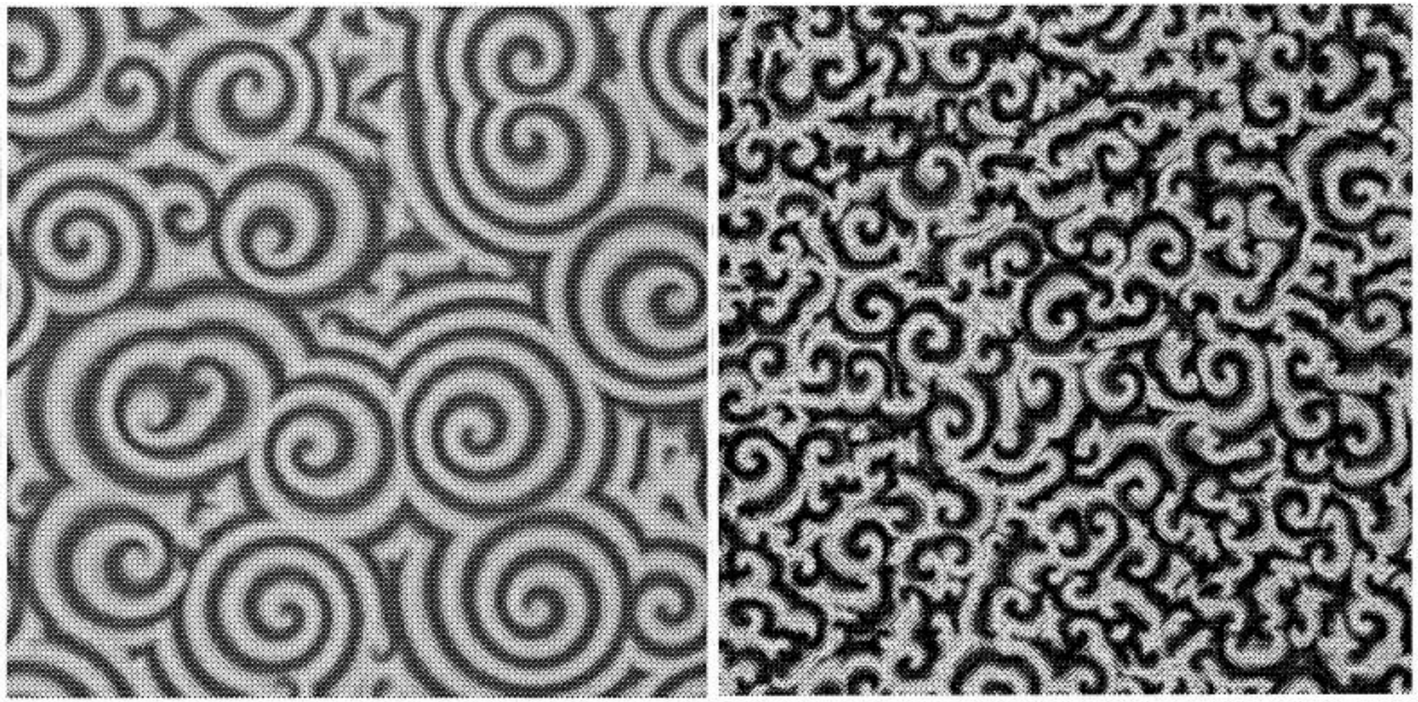
$$y_{n+1}(\mathbf{k}) = x_n(\mathbf{k})(1 - \exp(-\beta y_n(\mathbf{k})) + D_2 \nabla^2 y_n(r).$$

- A: periodic, $\lambda = 0$
- B: chaotic, $\lambda > 0$
- C: stable, $\lambda < 0$
- D: chaotic, $\lambda > 0$



Results – LE obtained from new method

- λ get from spatiotemporal method (Spiral waves showed by the host–parasitoid CML)
- λ = 0.013 (quasiperiodic) λ = 0.125 (chaotic)



$$x_{n+1}(\mathbf{k}) = \mu x_n(\mathbf{k})(1 - x_n(\mathbf{k})) \times \exp(-\beta y_n(\mathbf{k})) + D_1 \nabla^2 x_n(r)$$

$$y_{n+1}(\mathbf{k}) = x_n(\mathbf{k})(1 - \exp(-\beta y_n(\mathbf{k}))) + D_2 \nabla^2 y_n(r).$$

Further discussion on dimensionality

- Recall that d is the value it used to find neighbor points

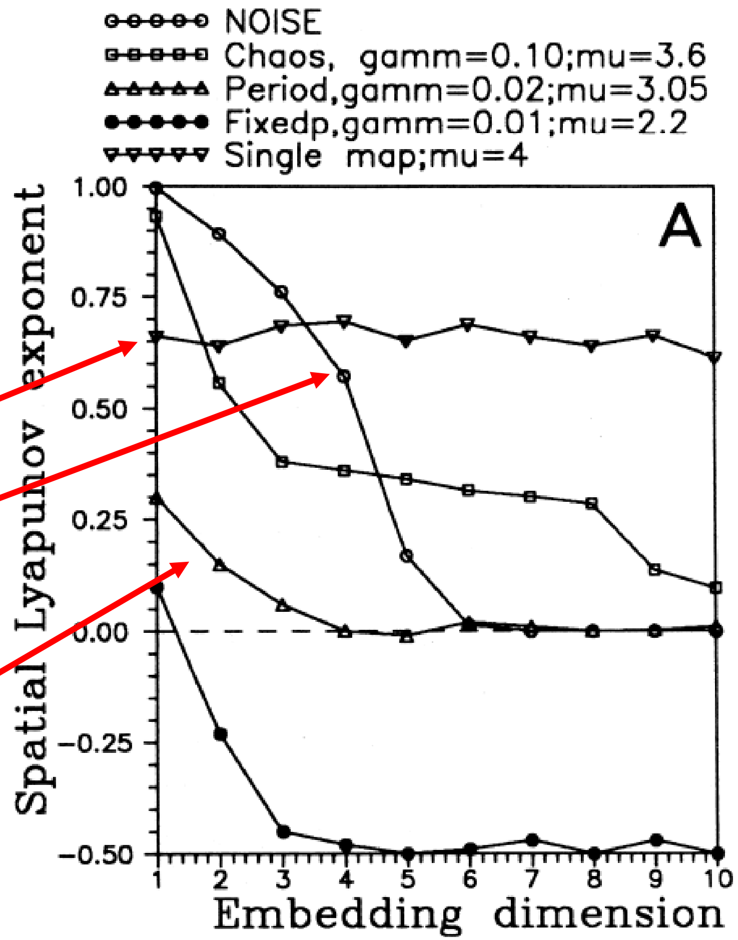
$$\| \mathbf{X}_i^j(\mathbf{k}) - \mathbf{X}_i^j(\mathbf{h}) \| = \left[\sum_{u=i}^{i+d-1} (x_u^j(\mathbf{k}) - x_u^j(\mathbf{h}))^2 \right]^{1/2} < \epsilon$$

- λ(d) shows a plateau after a certain d=d0
- Where d0 is an estimation of system's dimensionality
  - For single map (logistic map without coupled), its dimension = 1, so λ keeps almost constant when d increases (d = d0 = 1)
  - For noise, d0 = inf, so no plateau
  - For others, d0 = 3, so logistic CML has dimension around 3

$$x_{n+1}(\mathbf{k}) = \mu x_n(\mathbf{k})(1 - x_n(\mathbf{k})) + D\nabla^2 x_n(r),$$

with D the diffusion rate and

$$\nabla^2 x_n(r) = x_n(i-1, j) + x_n(i+1, j) + x_n(i, j-1) + x_n(i, j+1) - 4x_n(i, j)$$



For Logistic CML

Conclusion

- It detects the presence of chaos in very short temporal series with information in different spatial points.

- It used CML models to prove the validity of the method

- Furthermore, dimension information can be inferred from this method, and can be used for validity check

Reference

- [1]R. V. Solé and J. Bascompte, “Measuring chaos from spatial information,” Journal of theoretical biology, vol. 175, no. 2, pp. 139–147, 1995, doi: 10.1006/jtbi.1995.0126.
- [2]Lyapunov exponent [Online]. Available: https://en.wikipedia.org/wiki/Lyapunov_exponent
- [3]A. Wolf, J. B. Swift, H. L. Swinney, and J. A. Vastano, “Determining Lyapunov exponents from a time series,” Physica. D, vol. 16, no. 3, pp. 285–317, 1985, doi: 10.1016/0167-2789(85)90011-9.

Thank you!